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Equivariant cross sections of complex Stiefel manifolds

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Abstract

Let G be a finite group and let M be a unitary representation space of G. A solution to the existence problem of G-equivariant cross sections of the complex Stiefel manifold $W_k(M)$ of unitary k-frames over the unit sphere $S(M)$ is given under mild restrictions on G and on fixed point sets. In the case G is an even ordered group, some sufficient conditions for the existence of G-equivariant real frame fields on spheres with complementary G-equivariant complex structures are also obtained, improving earlier results about odd ordered groups. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let G be a topological group and M be a unitary representation space of G. Denote the unit sphere in M by $S(M)$. Then the tangent bundle $T(S(M))$ of $S(M)$ inherits a natural G-action. Let us denote the Stiefel manifold of unitary k-frames on M by $W_k(M)$. Let $\rho: W_k(M) \to S(M)$ be the projection map sending a unitary k-frame $(u_1, u_2, \dots, u_k)$ to u_k. Then $W_k(M)$ inherits the natural G-action

$$g(u_1, u_2, \dots, u_k) = (gu_1, gu_2, \dots, gu_n),$$

and ρ is equivariant with respect to this action.

In this paper we give necessary and sufficient conditions for the existence of a G-equivariant cross section of $W_k(M)$ for certain finite group actions, under mild fixed point conditions.

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When G acts trivially, this is answered in the work of Atiyah and Todd [3] and Adams and Walker [2] in 1960s. In this case M can be considered as $\mathbb{C}^n$. In [3] Atiyah and Todd gave necessary conditions for the existence of a cross section of $W_k(\mathbb{C}^n)$, and Adams and Walker proved in [2] that these conditions are also sufficient. The final solution is that $W_k(\mathbb{C}^n)$ admits a cross section if and only if a certain number b_k, the so-called Atiyah–Todd number, divides n. (The Atiyah–Todd number b_k is also called the kth complex James number; however the latter is now more commonly used in some other sense.) The number b_k can be given as follows:

Let $b_k = \prod p^{\nu_p(b_k)}$ be the prime decomposition of b_k where $\nu_p(b_k)$ denotes the largest integer r such that p^r divides b_k. Then

$$\nu_p(b_k) = \begin{cases} \max\{r + \nu_p(r)\colon\ 1 \leqslant r \leqslant \lfloor (k-1)/(p-1) \rfloor\} & \text{if } p \leqslant k, \\ 0 & \text{if } p > k. \end{cases}$$

A solution of the similar problem for the G-equivariant cross section of real Stiefel manifolds was given by Becker in 1972 [4] for free G-actions, and by Namboodiri [10] and Kakutani [8] in 1983 for actions with fixed points.

On the other hand, the solution of the problem when G acts freely on M is still open. The main difficulty in this case is that there are no explicitly known (non-equivariant) complex frame fields on odd dimensional spheres other than the complex 1-frame fields on S^{4n-1} that assign to each $x \in S^{4n-1}$ the complex frame (jx) or (kx) where j and k are the usual quaternionic units.

To give the main results of the paper we shall first define certain numbers and fix some notation.

Throughout the whole paper, we shall denote by $C(n; i_1, i_2, \ldots, i_r)$ with $i_1 + i_2 + \cdots + i_r = n$ the multinomial coefficient $n!/(i_1! \cdots i_r!)$. The binomial coefficient $C(n; i, (n-i))$ will be denoted by $C(n, i)$ as usual.

There exists a unique polynomial T_p with degree p and with the property that $T_p(x + x^{-1} - 2) = x^p + x^{-p} - 2$. It can be shown that $T_2(x) = 4x + x^2$, and for any odd integer $p = 2q + 1$

$$T_p(x) = x \left\{ \sum_{j=0}^{q} \frac{2q+1}{2j+1} C(q+j, 2j) x^j \right\}^2 \tag{1.1}$$

(see, e.g., [12]).

For positive integers l, p, and r with $p \geqslant 2$ and $r \geqslant l$ we define the integer $R_p(r,l)$ by

$$\bigl(T_p(x)\bigr)^l = \sum_{r=l}^{lp} R_p(r,l) x^r. \tag{1.2}$$

Thus when $p = 2q + 1$ is an odd integer with $q > 0$,

$$R_p(r,l) = \sum_{\substack{i_0 + i_1 + \cdots + i_q = 2l,\\ i_1 + 2i_2 + \cdots + q i_q = r - l}} p^{2l} C(2l; i_0, \ldots i_q) \left(\prod_{j=0}^{q} \frac{C(q+j, 2j)^{i_j}}{(2j+1)^{i_j}} \right), \tag{1.3}$$

where the sum is taken over nonnegative integers.

Next, for positive integers p, r, l, we shall define the integer $Q_p(r,l)$ to be the coefficient of x^r in the polynomial $((1+x)^p-1)^l$. Explicitly

$$Q_p(r,l)=\sum_{\substack{i_1+\cdots+i_p=l,\\ i_1+2i_2+\cdots+pi_p=r}} C(l;i_1,\ldots,i_p)\left(\prod_{j=1}^{p} C(p,j)^{i_j}\right). \tag{1.4}$$

Now let d, k be positive integers, p be a prime, τ be a permutation of $\{1,2,\ldots,d\}$, and $\mathcal{M}=(m_1,m_2,\ldots,m_d)$ be a d-tuple of nonnegative integers. For each pair j, r of integers with $1\leqslant j\leqslant d$ and $1\leqslant r$ we define rational numbers $O_r^j(\mathcal{M},\tau,p,k)$ recursively with respect to r as follows:

$$O_r^j(\mathcal{M},\tau,p,k)=\sum_{l=0}^{d-1} m_{\tau^l(j)}k^{2l}\quad\text{for } r=1,$$

and

$$O_r^j(\mathcal{M},\tau,p,k)=\sum_{l=\lceil r/k\rceil}^{r-1} R_k(r,l)S_l^{r-1}\left(\sum_{i=1}^{d}\frac{O_l^{\tau^i(j)}(\mathcal{M},\tau,p,k)}{p^{\nu_p(1-k^{2ld})}}k^{2r(i-1)}\right)$$
$$\text{for } 2\leqslant r,$$

where

$$S_l^s=\prod_{i=l+1}^{s}\frac{1-k^{2id}}{p^{\nu_p(1-k^{2id})}}$$

for any pair of non-negative integers s and l with $l<s$, $S_s^s=1$, and $\lceil n\rceil$ denotes the smallest integer greater than or equal to n.

Similarly, for each pair j, r of integers with $1\leqslant j\leqslant d$ and $1\leqslant r$, we define rational numbers $U_r^j(\mathcal{M},\tau,p,k)$ recursively with respect to r by

$$U_r^j(\mathcal{M},\tau,p,k)=\sum_{l=0}^{d-1} m_{\tau^l(j)}k^{l}\quad\text{for } r=1,$$

and

$$U_r^j(\mathcal{M},\tau,p,k)=\sum_{l=\lceil \frac{r}{k}\rceil}^{r-1} Q_k(r,l)T_l^{r-1}\left(\sum_{i=1}^{d}\frac{U_l^{\tau^i(j)}(\mathcal{M},\tau,p,k)}{p^{\nu_p(1-k^{ld})}}k^{r(i-1)}\right)\quad\text{for } 2\leqslant r,$$

where for any pair of positive integers l and s with $l<s$

$$T_l^s=\prod_{i=l+1}^{s}\frac{1-k^{\mathrm{id}}}{p^{\nu_p(1-k^{\mathrm{id}})}},$$

and $T_s^s=1$. Note that for any given r, when $O_l^j(\mathcal{M},\tau,p,k)$ is a multiple of $p^{\nu_p(1-k^{2ld})}$ for each l and j with $\lceil r/k\rceil\leqslant l\leqslant r-1$, $1\leqslant j\leqslant d$, then $O_r^j(\mathcal{M},\tau,p,k)$ is an integer. Similarly when $U_l^j(\mathcal{M},\tau,p,k)$ is a multiple of $p^{\nu_p(1-k^{ld})}$ for each l and j with $\lceil r/k\rceil\leqslant l\leqslant r-1$ and $1\leqslant j\leqslant d$, then $U_r^j(\mathcal{M},\tau,p,k)$ is an integer.

Given a finite group G we recall that a real G-module V is said to be of type $\mathbb{R}$ (respectively $\mathbb{C}$, $\mathbb{H}$) if its endomorphism algebra $\mathrm{Hom}_G(V, V)$ is $\mathbb{R}$ (respectively $\mathbb{C}$, $\mathbb{H}$).

There is a similar terminology for complex G-modules [5]. Let V be a complex G-module. A real structure on V is a real conjugate-linear map G-map $J: V \to V$ such that $J^2 = \mathrm{id}$. A quaternionic structure on V is a conjugate linear G-map $J: V \to V$ such that $J^2 = -\mathrm{id}$. In both cases J is called the structure map. A complex representation is said to be of real (respectively quaternionic type) if it admits a real (respectively quaternionic) structure. If V is of either of these types, structure map provides an isomorphism $V \cong \overline{V}$, where $\overline{V}$ is the conjugate module of V. If $V \ncong \overline{V}$, then the complex G-module is of complex type. We call a real (complex) G-module M of type $\mathbb{R}$ if it is a sum of real (complex) type-$\mathbb{R}$ modules. We define type-$\mathbb{C}$ and type-$\mathbb{H}$ G-modules (real or complex) similarly.

Now we can state our main theorem. Let p be a prime and G be a p-group with no type-$\mathbb{H}$ irreducibles (e.g., an Abelian p-group). Let k be an odd generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$, which will be taken as 3 if $p = 2$. We recall that the realification $r(V)$ of every irreducible complex G-module of type $\mathbb{C}$ is a real irreducible module of type $\mathbb{C}$ and the realification of every complex irreducible G-module V of type $\mathbb{R}$ can be written as $r(V) = U \oplus U$, where U is a real irreducible module of type $\mathbb{R}$ (see [5]).

By Lemma 6.1 of [10], the Adams operation $\psi^k_{\mathbb{R}}$ permutes real irreducible G modules preserving their types, and since realification commutes with Adams operations, all complex irreducible G-modules are permuted by $\psi^k_{\mathbb{C}}$ preserving their types. Let $\mathcal{O}_1$, $\mathcal{O}_2, \ldots, \mathcal{O}_{s_1}$ be the orbits consisting of complex irreducible G-modules of type $\mathbb{R}$, and let us denote by $\mathcal{O}_{s_1+1}, \mathcal{O}_{s_1+2}, \ldots, \mathcal{O}_{s_1+s_2}$ the orbits consisting of complex irreducible G-modules of type $\mathbb{C}$. For each fixed i with $1 \leqslant i \leqslant s_1 + s_2$, let $\{V_{i,j}\}$ be the set of irreducibles constituting $\mathcal{O}_i$ with $1 \leqslant j \leqslant d_i$. Let τ_i denote the inverse of the permutation determined by the action of $\psi^k_{\mathbb{C}}$ on the orbit $\mathcal{O}_i$ and let $M = \sum_{i,j} m_{i,j} V_{i,j}$ be a unitary representation space of G where each $m_{i,j} \geqslant 0$. Finally let us write $\mathcal{M}_i = (m_{i,1}, m_{i,2}, \ldots, m_{i,d_i})$. Then we have

Theorem 1.1. *If the Stiefel manifold $W_n(M)$ admits a G-cross section then the following conditions are satisfied*:

(i) $p^{\nu_p(1-k^{2rd_i})} \mid O^j_r(\mathcal{M}_i, \tau_i, p, k)$ *for each* $1 \leqslant i \leqslant s_1, 1 \leqslant j \leqslant d_i$ *and* $1 \leqslant r \leqslant t$, *where* $t = m$ *if* $n = 2m + 1$, $t = 2m$ *if* $n = 4m + 2$, *and* $t = 2m + 1$ *if* $n = 4m + 4$ *for some integer* $m \geqslant 0$. *Moreover, in the case* $n = 4m + 2$ *and* $p = 2$, $O^j_{2m+1}(\mathcal{M}_i, \tau_i, 2, 3)$ *is even for each* $1 \leqslant i \leqslant s_1$ *and* $1 \leqslant j \leqslant d_i$.

(ii) $p^{\nu_p(1-k^{rd_i})} \mid U^j_r(\mathcal{M}_i, \tau_i, p, k)$ *for each* $s_1 + 1 \leqslant i \leqslant s_1 + s_2$, *for each* $1 \leqslant j \leqslant d_i$, *and for all* $1 \leqslant r \leqslant n - 1$.

(iii) *For any prime* $q \neq p$, $q^{\nu_q(b_n)} \mid \dim_{\mathbb{C}} M^H$ *for all* $H < G$, *where* b_n *denotes the* n*th Atiyah–Todd number and* M^H *is the* H*-fixed point set of* M.

If G is an Abelian p-group and $\dim_{\mathbb{R}} M^G \geqslant 4n - 2$, *then the conditions* (i)–(iii) *imply that $W_n(M)$ admits a G-cross section.*

In order to state the results for arbitrary finite groups, we require the notion of the "excisive universe" [10].

If W is a G-module, let W^∞ be direct sum of infinitely many copies of W. If $\{W_i\}$ is a collection of G-modules containing the trivial G-module, we say that the space $\bigoplus_i W_i^\infty$, topologized as the colimit of its finite-dimensional subspaces, is a universe.

Given a G-module M, there is a G-universe $\mathcal{U}(M)$ which makes desuspension possible in several instances (again see [10]). Let $\mathrm{Gap}(M)$ be the set of G-modules V such that $\dim_{\mathbb{R}} M^K > \dim_{\mathbb{R}} M^H$ whenever $K < H$ and $\dim_{\mathbb{R}} V^K > \dim_{\mathbb{R}} V^H$, where in general V^H denotes the H-fixed point set of V for any G-module V and for $H < G$. The G-universe $\mathcal{U}(M)$ is now defined by

$$\mathcal{U}(M) = \bigoplus_{W_i \in \mathrm{Gap}(M) \cap \mathrm{Irr}(G)} W_i^\infty,$$

where $\mathrm{Irr}(G)$ is the set of irreducible G-modules. When $\mathcal{U}(M)$ has enough orbit types (for instance, all of it), we say that $\mathcal{U}(M)$ is an excisive universe. This will be made precise in Section 4. For any G-space X we shall denote the set of all isotropy groups of points of X by $\mathrm{Iso}(X)$. We shall denote by WH the quotient $N_G(H)/H$ when $H < G$.

The following theorem reduces the problem for arbitrary finite groups to the p-group case.

Theorem 1.2. *Let G be a finite group and let M be a unitary G-module. For every prime p, let*

$$\Phi(p) = \bigl\{H < G\colon\ (|WH|, p) = 1\bigr\},$$

and for any $H < G$ let H_p be the smallest normal subgroup of H with p-primary index. If $W_n(M)$ admits a G-equivariant cross section then the Stiefel manifold $W_n(M^{H_p})$ admits an H/H_p-equivariant cross section for every prime p and for all $H \in \Phi(p)$.

The converse is also true if, in addition, $\dim_{\mathbb{R}} M^G \geqslant 4n - 2$ when $n > 1$, and $\mathcal{U}(M)$ is an excisive universe.

The theorems and lemmas leading to the proofs of Theorems 1.1 and 1.2 yield some sufficient conditions for the existence of equivariant frame fields on spheres with complementary equivariant complex structures. We get results for Abelian groups of even order. Previously this was solved for all odd groups and for $\mathbb{Z}/2\mathbb{Z}$ [11].

Let us recall certain definitions and terminology from [11]. Let G be a finite group and M be a unitary representation space of G. For an integer k, a G equivariant k-frame field on $S(M)$ is an ordered set of real k linearly independent vector fields each of which is G-equivariant. Let $u(x) = (u_1(x), u_2(x), \ldots, u_k(x))$ be a frame field, let η denote the orthogonal complement of the subbundle $T(S(M))$ spanned by $u_1, u_2, \ldots, u_k$, and let $E(\eta)$ denote its total space. We call u a k-frame field with complementary G-equivariant complex structure if there exists a G-bundle map $I: E(\eta) \to E(\eta)$ such that $I^2 = -1$. Thus, if we denote by I_x the restriction of I to the fibre η_x of η over x, then $I_{gx}(gv) = gI_x(v)$ for every x in the unit sphere $S(M)$ and v in η_x.

Theorem 1.3. *Let G be an Abelian 2-group and let M be a unitary representation space of G such that $\dim_{\mathbb{R}} M^G \geqslant 4n-2$ when $n > 1$. Write $M = \sum_{i,j} m_{i,j} V_{i,j}$ with each $m_{i,j} \geqslant 0$. Let $\mathcal{M}_i, \tau_i, d_i$ be as in Theorem 1.1, and assume the following*

(i) $2^{\nu_2(1-3^{2rd_i})} \mid O_r^j(\mathcal{M}_i, \tau_i, 2, 3)$ *for each $1 \leqslant i \leqslant s_1$, $1 \leqslant j \leqslant d_i$ and $1 \leqslant r \leqslant t$, where $t = m$ if $n = 2m+1$, $t = 2m$ if $n = 4m+2$, and $t = 2m+1$ if $n = 4m+4$ for some integer $m \geqslant 0$. Moreover, in case $n = 4m+2$, assume $O_{2m+1}^j(\mathcal{M}_i, \tau_i, 2, 3)$ is even.*

(ii) $2^{\nu_2(1-3^{rd_i})} \mid U_r^j(\mathcal{M}_i, \tau_i, 2, 3)$ *for each $s_1 + 1 \leqslant i \leqslant s_1 + s_2$, for each $1 \leqslant j \leqslant d_i$, and for all $1 \leqslant r \leqslant n-1$.*

Then, there exists a (*real*) *G-equivariant $(2n-1)$-frame field on $S(M)$ with a complementary G-equivariant complex structure.*

We remark that the answer to the converse is also known for odd group actions and for $G = \mathbb{Z}/2\mathbb{Z}$ [11]. However, the converse for other groups requires further study.

As in the case of G-equivariant cross sections of $W_n(M)$, under the excisive universe hypothesis, the study of the existence of G-equivariant frame fields with complementary complex structures for even-ordered groups G is reduced to the 2-group case.

Theorem 1.4. *Let G be a group of even order and let M be an unitary representation space of G. If there exists a* (*real*) *G-equivariant $(2n-1)$-frame field on $S(M)$ with a complementary G-equivariant complex structure, then $S(M^{H_2})$ admits an H/H_2-equivariant $(2n-1)$-frame field with a complementary G-equivariant complex structure for each $H \in \Phi(2)$.*

The converse is also true if, in addition, $\dim(M)^G \geqslant 4n-2$ *when $n > 1$, and $\mathcal{U}(M)$ is excisive.*

In Section 2, we reduce the problem to determining when the realification $r(M \otimes (\xi - 1))$ of the bundle $M \otimes (\xi - 1)$, where ξ is the Hopf line bundle on $\mathbb{C}P^{n-1}$, vanishes in the equivariant J-group of $\mathbb{C}P^{n-1}$. In Section 3, we determine when this bundle is in the kernel of the localized homomorphism $J_{(p)}$ in the case G is a p-group. The results for the general case and the proofs of main theorems are all given in Section 4.

2. Reduction of the problem to a question in equivariant J-groups of complex projective spaces

We recall some definitions from [10]. Let G be a finite group and let M be a G-module. Let $\mathcal{U}(M)$ be the G-universe defined in the introduction. In this section we shall write $\mathcal{U}$ instead of $\mathcal{U}(M)$ when M is understood from the context. If $H < G$ then $\mathcal{U}(M)$ considered as an H-space will be an H-universe. We shall denote this universe by $\mathcal{U}(M, H)$ or simply by $\mathcal{U}(H)$ when M is understood. Let X be a G-space such that the set $\mathrm{Iso}(X)$ of all isotropy groups of X is a subset of all isotropy subgroups $\mathrm{Iso}(\mathcal{U})$ of $\mathcal{U}$. Then we define

$$KO_G(X; \mathcal{U}) = \left\{ E - F \in KO_G(X)\colon\ E_x, F_x \subset \mathcal{U}(M, G_x) \text{ for each } x \in X \right\}$$

where inclusion here is understood as inclusion up to isomorphism. Let

$$TO_G(X;\mathcal{U}) = \big\{E - F \in KO_G(X;\mathcal{U}):\ S(E \oplus \boldsymbol{V}) \sim_G S(F \oplus \boldsymbol{V}) \\ \text{for some } V \in \mathrm{Gap}(M)\big\},$$

where, in general $E \sim_G F$ for G-fibre bundles E and F means that E and F are G-fibre homotopy equivalent, and $\boldsymbol{V}$ denotes the product bundle $X \times V$ for any G-module V. Define the group $JO_G(X;\mathcal{U})$ to be the quotient $KO_G(X;\mathcal{U})/TO_G(X;\mathcal{U})$.

The main result of this section is the following theorem

Theorem 2.1. *Let G be a finite group and M be a unitary representation space of G. Let $\rho: W_n(M) \to S(M)$ have a G-cross section. Then, the realification $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ of $\boldsymbol{M} \otimes (\xi - 1)$ vanishes in $JO_G(\mathbb{C}P^{n-1})$, where ξ is the Hopf line bundle over $\mathbb{C}P^{n-1}$ and both $\mathbb{C}P^{n-1}$ and ξ are considered with trivial G action.*

Conversely, if $\dim M^G \geqslant 4n - 2$ for $n > 1$ and if $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in $JO_G(\mathbb{C}P^{n-1};\mathcal{U})$, then ρ has a G-cross section.

To prove this theorem we first prove the following desuspension theorem.

Theorem 2.2. *Let M be a unitary representation space of G and assume that $\dim_{\mathbb{R}} M^G \geqslant 4n - 2$ when $n > 1$. Further assume that $V \subset \mathcal{U}$ is a G-module with $S(\mathrm{r}(\boldsymbol{M} \otimes \xi) \oplus \boldsymbol{V}) \sim_G S(r(\boldsymbol{M}) \oplus \boldsymbol{V})$. Then $S(\mathrm{r}(\boldsymbol{M} \otimes \xi)) \sim_G S(r(\boldsymbol{M}))$.*

Proof. The proof is similar to the proof of Corollary 5.1 of [10], proved for real projective space. Mod p version of this result for $\mathbb{C}P^{n-1}$ is proved in Lemma 2.4 of [11] (we would like to note that there is a misprint in the statement of that lemma, λ must be an integer prime to p, not just an integer). We need to modify the proof slightly; in fact, the proof is easier in our case. The main tool in proving this theorem is Lemma 3.3 of [11] and we shall need it in the special case $\lambda = 1$. Our hypotheses guarantee that conditions (1)–(4) of Lemma 3.3 of [11] are satisfied. So, we are done if we can prove that

$$\dim_{\mathbb{R}} M^K - \dim_{\mathbb{R}} M^H > 2n - 1$$

whenever V is a G-module, $H < K$, and $\dim_{\mathbb{R}} V^K > \dim_{\mathbb{R}} V^H$. By restriction to H-fixed point sets, the hypothesis $S(\mathrm{r}(\boldsymbol{M} \otimes \xi) \oplus \boldsymbol{V}) \sim_G S(\mathrm{r}(\boldsymbol{M}) \oplus \boldsymbol{V})$ implies that $(\dim_{\mathbb{R}} M^H/2)\mathrm{r}(\xi - 1) = 0$ in $JO(\mathbb{C}P^{n-1})$; so $2b_n$ divides $\dim_{\mathbb{R}} M^H$, where b_n denotes Atiyah–Todd number given in the introduction. Similarly $2b_n$ divides $\dim_{\mathbb{R}} M^K$. Hence $2b_n$ divides $\dim_{\mathbb{R}} M^K - \dim M^H$. Since $\dim_{\mathbb{R}} M^K - \dim_{\mathbb{R}} M^H > 0$ by hypothesis, it suffices to prove that $2b_n > 2n - 1$ to complete the proof.

Since $b_2 = 2$ and $b_3 = 24$, clearly we have $2b_n > 2n - 1$ for $1 \leqslant n \leqslant 3$.

For $n \geqslant 4$, using the explicit the value of $\nu_2(b_n)$ given in the introduction, we have

$$2b_n \geqslant 2^{\nu_2(b_n)+1} > 2^{\nu_2(b_n)} \geqslant 2^{n-1} > 2n - 1.$$

So we have $2b_n > 2n - 1$ in any case, and hence by Lemma 3.4 of [11] there exists a fibrewise G-map between $S(\mathrm{r}(\boldsymbol{M} \otimes \xi))$ and $S(\mathrm{r}(\boldsymbol{M}))$ that has degree one on all fibres. By the Equivariant Dold Theorem [14, 1.11] the result follows. □

Proof of Theorem 2.1. This is very similar to the proof of Theorem 3.6 in [11]. In fact, it is much simpler; instead of Lemma 3.4 of that paper, Theorem 2.2 above should be used, and instead of the Equivariant Dold Theorem Mod k [7] and its converse (Theorem 2.1 of [11]), it suffices to use Equivariant Dold Theorem [14, 1.11]. □

3. Localization at p, p-group case

To state and prove the main results of this section, we begin by recalling some basic definitions. Let G be a finite group and let M be a G-module. Throughout this section $\mathcal{U}$ will denote $\mathcal{U}(M)$ unless otherwise specified. Let us denote by $R(\mathbb{R};\mathcal{U})$ the Grothendieck group of type-$\mathbb{R}$ G-modules in $\mathrm{Gap}(M)$. We define $R(\mathbb{C};\mathcal{U})$ and $R(\mathbb{H};\mathcal{U})$ similarly.

Then we can deduce from [13] the isomorphism

$$\begin{aligned} KO_G(\mathbb{C}P^n;\mathcal{U}) \cong\ & KO(\mathbb{C}P^n)\otimes R(\mathbb{R};\mathcal{U})\oplus KU(\mathbb{C}P^n)\otimes R(\mathbb{C};\mathcal{U}) \\ & \oplus KSp(\mathbb{C}P^n)\otimes R(\mathbb{H};\mathcal{U}). \end{aligned} \tag{3.1}$$

For future use, when $w=\sum_{j=1}^{d} c^j(v)\otimes \boldsymbol{V}_j$ is an element of $KU(\mathbb{C}P^n)\otimes R(\mathbb{C})$, the element corresponding to w in $KO_G(\mathbb{C}P^n;\mathcal{U})$ under the isomorphism is just the realification of w (e.g., see [4]).

Let E, F be two vector bundles in $KO_G(X;\mathcal{U})$. Then we recall [6,10] that $E\sim_{\mathrm{loc}} F$ if for each $H<G$, there exists a G-module $V(H)\subset\mathcal{U}$ and G-maps

$$f: S\bigl(E\oplus \boldsymbol{V}(\boldsymbol{H})\bigr)\to S\bigl(F\oplus \boldsymbol{V}(\boldsymbol{H})\bigr) \quad\text{and}\quad g: S\bigl(F\oplus \boldsymbol{V}(\boldsymbol{H})\bigr)\to S\bigl(E\oplus \boldsymbol{V}(\boldsymbol{H})\bigr)$$

such that f_x^H and g_x^H are homotopy equivalences for each $x\in X$. Let

$$TO_G^{\mathrm{loc}}(X;\mathcal{U})=\bigl\{E-F\in KO_G(X;\mathcal{U}):\ E\sim_{\mathrm{loc}} F\bigr\}.$$

We define

$$JO_G^{\mathrm{loc}}(X;\mathcal{U})=KO_G(X;\mathcal{U})/TO_G^{\mathrm{loc}}(X;\mathcal{U}).$$

Now we have

Theorem 3.1. *Let $k\in\mathbb{Z}$ be an odd generator of the group of units of $\mathbb{Z}/p^2\mathbb{Z}$. Let G be a p-group with no type $\mathbb{H}$ irreducibles. Then $J\circ(1-\psi^k)=0$ in the sequence*

$$KO_G(\mathbb{C}P^n;\mathcal{U})_{(p)} \xrightarrow{1-\psi^k} KO_G(\mathbb{C}P^n;\mathcal{U})_{(p)} \xrightarrow{J} JO_G^{\mathrm{loc}}(\mathbb{C}P^n;\mathcal{U})_{(p)}.$$

When G is an Abelian p-group the sequence is exact.

We need some lemmas and propositions to prove this theorem.

Lemma 3.2. *Suppose V is an irreducible orthogonal real two-dimensional G-module of type $\mathbb{C}$, and let l be a positive integer. Then there exists a $G\times S^1$-map $f:S(V)\to S(\psi^k(V))$ that has degree prime to p, where S^1 acts on $S(V)$ by $\mu\cdot x=\bar{\mu}^l\cdot x$, and it acts on $S(\psi^k(V))$ by $\mu\cdot x=\bar{\mu}^{kl}\cdot x$. Here $\bar{\mu}$ denotes the complex conjugate of μ, and the multiplications on the right sides of both equalities are complex multiplications.*

Proof. Since k is odd we have the O(2)-map $f : S(V) \to S((\psi^k(V))$ which is simply $z \to z^k$ (see [1]). This is G-equivariant since G acts unitarily, and it is S^1-equivariant with respect S^1-actions given on $S(V)$ and $S(\psi^k(V))$ by definition. □

Proposition 3.3. *Let ξ be the canonical Hopf line bundle on $\mathbb{C}P^n$ and let l be a positive integer. Let E be a bundle either of the form $\mathrm{r}(\xi)^l \otimes \boldsymbol{U}$, where U is a real one-dimensional G-module of type $\mathbb{R}$, or of the form $\mathrm{r}(\xi^l \otimes \boldsymbol{V})$, where V is a real irreducible two-dimensional G-module of type $\mathbb{C}$. Then there exists a fiberwise G-map $f : S(E) \to S(\psi^k(E))$ such that f_x has degree prime to p for each $x \in \mathbb{C}P^n$.*

Proof. First assume that E is a G-vector bundle of the form $\mathrm{r}(\xi^l \otimes \boldsymbol{V})$, where $\boldsymbol{V}$ is an irreducible real two-dimensional G-module of complex type. The total space of the bundle $\xi^l \otimes \boldsymbol{V}$ is of the form $(S^{2n-1} \times V)/S^1$, where S^1 acts on $S^{2n-1} \cong S(\mathbb{C}^n)$ by complex multiplication and on V by $\mu \cdot y = \bar{\mu}^l \cdot y$. On the other hand, the total space of the bundle $\psi^k(E)$ is given by $(S^{2n-1} \times \psi^k(V))/S^1$, where S^1 acts on $\psi^k(V)$ by $\mu \cdot y = \bar{\mu}^{kl} \cdot y$. Now the map given in Lemma 3.2 induces the desired map f.

Next assume that $E = \mathrm{r}(\xi)^l \otimes \boldsymbol{U}$, where $\boldsymbol{U}$ is an irreducible real one-dimensional G-module of real type, and l is odd. Let c denote the complexification map. Keeping in mind that $\xi\bar{\xi}$ is trivial, we have

$$\begin{aligned}
\mathrm{c}\big(\mathrm{r}(\xi)^l \otimes \boldsymbol{U}\big) &= \big(\mathrm{cr}(\xi)\big)^l \otimes \mathrm{c}(\boldsymbol{U}) = (\xi + \bar{\xi})^l \otimes \mathrm{c}(\boldsymbol{U}) = \left(\sum_{i=0}^{l} \binom{l}{i} \xi^{l-i} \bar{\xi}^i\right) \otimes \mathrm{c}(\boldsymbol{U}) \\
&= \left(\sum_{2i<l} \binom{l}{i} \xi^{l-i} \bar{\xi}^i + \sum_{2i<l} \binom{l}{i} \xi^{i} \bar{\xi}^{l-i}\right) \otimes \mathrm{c}(\boldsymbol{U}) \\
&= \sum_{2i<l} \binom{l}{i} (\xi^{l-2i} + \bar{\xi}^{l-2i}) \otimes \mathrm{c}(\boldsymbol{U}) = \sum_{2i<l} \binom{l}{i} \mathrm{cr}(\xi^{l-2i}) \otimes \mathrm{c}(\boldsymbol{U}) \\
&= \mathrm{c}\left(\sum_{2i<l} \binom{l}{i} \mathrm{r}(\xi^{l-2i}) \otimes \boldsymbol{U}\right).
\end{aligned}$$

If, on the other hand, l is even, there exists an extra term corresponding to $i = l/2$, namely $\mathrm{c}(\binom{l}{l/2}\mathrm{r}(1) \otimes \boldsymbol{U})$. Hence $\mathrm{r}(\xi)^l \otimes \boldsymbol{U}$ is a sum of bundles of the form $\mathrm{r}(\xi^j) \otimes \boldsymbol{U}$. This follows from the fact that c is injective when $n = 2$ and $n = 4m + 3$. If $n = 4m + 1$, we get the same result through the epimorphism $i^* : KO(\mathbb{C}P^{4m+2}) \to KO(\mathbb{C}P^{4m+1})$, where $i : \mathbb{C}P^{4m+1} \to KO(\mathbb{C}P^{4m+2})$ is the inclusion. Thus it suffices to prove the result for a bundle of the form $\mathrm{r}(\xi^j) \otimes \boldsymbol{U}$ which is just $\mathrm{r}(\xi^j \otimes \mathrm{c}(\boldsymbol{U}))$. The proof in this case is exactly the same as the previous case, as now $\mathrm{c}(\boldsymbol{U})$ is an irreducible real two-dimensional G-module of type $\mathbb{C}$. □

For the next lemma we recall the following (see [10]). If E and F are vector bundles in $KO_G(X; \mathcal{U})$, we write $E \sim_p F$ if there exists $V \subset \mathcal{U}$ and fibrewise G-maps

$$f : S(E \oplus \boldsymbol{V}) \to S(F \oplus \boldsymbol{V}) \quad \text{and} \quad g : S(F \oplus \boldsymbol{V}) \to S(G \oplus \boldsymbol{V})$$

such that f_x, g_x have degree prime to p for each $x \in X$. Let

$$TO_G(X;\mathcal{U};\,p) = \left\{E - F \in KO_G(X;\mathcal{U}):\ E \sim_p F\right\}$$

and define $JO_G(X;\mathcal{U};\,p)$ to be the quotient $KO_G(X;\mathcal{U};\,p)/TO_G(X;\mathcal{U};\,p)$. We have

Lemma 3.4. *Let G be an Abelian p-group. Then we have*

$$(1-\psi^k)\big(KO_G(\mathbb{C}P^n;\mathcal{U})\big)_{(p)} \subset TO_G(\mathbb{C}P^n;\mathcal{U};\,p)_{(p)}.$$

Proof. Since G is a finite Abelian group, every irreducible real G-module is either one-dimensional and of real type, or two-dimensional and of complex type (e.g., see Proposition 8.8 of [5]). Thus every element of $KO_G(\mathbb{C}P^n;\mathcal{U})$ is a sum and difference of bundles which are of the form mentioned in the statement of the Proposition 3.3 and some trivial bundles. The map f we constructed in Proposition 3.3 has the property that f_x has degree prime to p for each $x \in \mathbb{C}P^n$. By exactly the same proof as that of Lemma 11.4.3 of [6], there is a map g in the reverse direction, having the property that g_x has degree prime to p for each $x \in \mathbb{C}P^n$. □

Lemma 3.5. *The canonical quotient map*

$$JO_G^{\mathrm{loc}}(\mathbb{C}P^n;\mathcal{U})_{(p)} \to JO_G(\mathbb{C}P^n;\mathcal{U};\,p)_{(p)}$$

is an isomorphism.

Proof. For $\mathbb{R}P^n$ and $p = 2$, this is proved in Lemma 6.5 of [10]. The proof is similar for $\mathbb{C}P^n$ and for an arbitrary prime. □

Proof of Theorem 3.1. First $\mathrm{Ker}J \subset \mathrm{Im}(1-\psi^k)$ holds since this is true when $\mathcal{U}$ is GR^∞, the universe generated by the regular representation of G. The other inclusion follows from Lemmas 3.4 and 3.5. □

Now, let G be a p-group with no type-$\mathbb{H}$ irreducibles, k be an odd generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$. Let us consider a unitary representation space M. By the results of the previous section, the cross section problem of $W_n(M)$ was reduced to determining when $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in $JO_G(\mathbb{C}P^{n-1})$, where ξ is the Hopf line bundle over $\mathbb{C}P^{n-1}$. Since we deal with groups G with no type-$\mathbb{H}$ irreducibles, we can write $M = M_1 \oplus M_2$, where M_1 is of type $\mathbb{R}$, M_2 is of type $\mathbb{C}$. Since ψ^k permutes the irreducibles of G preserving their types, the problem is reduced to determining when $\mathrm{r}(\boldsymbol{M}_1 \otimes (\xi - 1))$ and $\mathrm{r}(\boldsymbol{M}_2 \otimes (\xi - 1))$ vanish in $JO_G(\mathbb{C}P^{n-1})$ simultaneously.

We recall that $KO(\mathbb{C}P^{n-1})$ is a truncated polynomial ring over the integers generated by y with the following relations

$$\begin{aligned}
&y^{m+1} = 0 && \text{if } n = 2m+1,\ m \geqslant 0;\\
&2y^{2m+1} = 0,\ y^{2m+2} = 0 && \text{if } n = 4m+2,\ m \geqslant 0;\\
&y^{2m+2} = 0 && \text{if } n = 4m+4,\ m \geqslant 0.
\end{aligned}$$

Similarly $KU(\mathbb{C}P^{n-1})$ is a truncated polynomial ring $\mathbb{Z}[v]/\langle v^n\rangle$ with $v=\xi-1$, where ξ is the Hopf line bundle (see [2]).

Let $V_1, V_2, \ldots, V_d$ be irreducible complex G-modules of type $\mathbb{R}$, in some orbit of the action of $\psi^k_{\mathbb{C}}$. Note that, because of the definition of $\mathcal{U}(M)$, and in view of the fact that $\dim_{\mathbb{R}} \psi^k(V)^H = \dim_{\mathbb{R}} V^H$ for each $H < K$ [10, Lemma 6.1], either all irreducible V_i in the orbit have realifications in $\mathcal{U}(M)$ or none of them satisfies that. Let τ be the inverse of the permutation determined by the action of $\psi^k_{\mathbb{C}}$ on $V_1, V_2, \ldots, V_d$. Then we have

Theorem 3.6. *Let $\mathcal{U}$ be either the universe GR^∞, the universe generated by the regular representation of G, or $\mathcal{U}(M)$, and let $z=\mathrm{r}(\boldsymbol{N}\otimes(\xi-1))$ where $\boldsymbol{N}=n_1V_1+\cdots+n_dV_d$ with $n_i \geqslant 0$ for each i. In case $\mathcal{U}=\mathcal{U}(M)$, assume $V_1, V_2, \ldots, V_d$ have realifications in $\mathcal{U}(M)$. Let us write $\mathcal{N}=(n_1,\ldots,n_d)$. If z vanishes in $JO_G(\mathbb{C}P^{n-1};\mathcal{U})_{(p)}$, then the following conditions are satisfied*:

(i) *$p^{\nu_p(1-k^{2rd})} \mid O_r^j(\mathcal{N},\tau,p,k)$ for each $1\leqslant j\leqslant d$ and for all $1\leqslant r\leqslant t$, where $t=m$ if $n=2m+1$, $t=2m$ if $n=4m+2$, and $t=2m+1$ if $n=4m+4$.*

(ii) *If $n=4m+2$ and $p=2$, $O^j_{2m+1}(\mathcal{N},\tau,2,3)$ is even for each $1\leqslant j\leqslant d$.*

If G is an Abelian p-group, these conditions are also sufficient.

Proof. If z vanishes in $JO_G(\mathbb{C}P^{n-1};\mathcal{U})_{(p)}$, then it vanishes in $JO_G^{\mathrm{loc}}(\mathbb{C}P^{n-1};\mathcal{U})_{(p)}$ by definitions. Conversely if z vanishes in $JO_G^{\mathrm{loc}}(\mathbb{C}P^{n-1};\mathcal{U})_{(p)}$, then it vanishes in $JO_G(\mathbb{C}P^{n};\mathcal{U})_{(p)}$ by the argument right after the proof of Theorem 6.6 of [10]. So the problem is reduced to determining when z is in the image of

$$1-\psi^k : KO_G(\mathbb{C}P^{n-1})_{(p)} \to KO_G(\mathbb{C}P^{n-1})_{(p)}.$$

Let $y=\mathrm{r}(\xi_{n-1}(\mathbb{C}))-2\in KO(\mathbb{C}P^{n-1})$, where $\mathrm{r}(\xi_{n-1})$ is the realification of the Hopf line bundle over $\mathbb{C}P^{n-1}$. We have $\psi^k(y)=T_k(y)$, where $T_k(y)$ is as in the introduction (see [12]). Then $\psi^k(y^l)=T_k(y)^l$ and hence, by Eq. (1.2),

$$\psi^k(y^l)=\sum_{r=l}^{kl} R_k(r,l)y^r. \tag{3.2}$$

We would like to find an element $w\in KO_G(\mathbb{C}P^{n-1})$ and an integer s prime to p such that

$$(1-\psi^k)(w)=sz, \tag{3.3}$$

where z is the realification $\mathrm{r}(\boldsymbol{N}\otimes(\xi-1))$ of $\boldsymbol{N}\otimes(\xi-1)$. Since each V_i is a complex irreducible G-module of type $\mathbb{C}$, we can write $V_i=U_i\oplus U_i$, where U_i is a real G-module of type $\mathbb{R}$. Let us write $\widetilde{\boldsymbol{N}}=n_1U_1+\cdots+n_dU_d$.

First we observe that $\mathrm{r}(\boldsymbol{N}\otimes(\xi-1))=\widetilde{\boldsymbol{N}}\otimes\mathrm{r}(\xi-1)=\widetilde{\boldsymbol{N}}\otimes y$. Second, in (3.3) s can be replaced by ss' for any fixed integer s' prime to p (while s is still one of the parameters that we are going to play with). In particular, we can take s' as

$$S_0^{t'}=\prod_{i=1}^{t'}\big((1-k^{2id})/p^{\nu_p(1-k^{2id})}\big)$$

given in the introduction, where $t' = t$ if $n = 2m + 1$ or $n = 4m + 4$ and $t' = t + 1$ if $n = 4m + 2$. The reason for this choice will be clear in the course of the proof. Put $S = S_0^{t'}$. Finally, in view of (3.1), we can write $w = \sum_{j=1}^{d} c^j(y) \otimes \boldsymbol{U}_j$, where $c^j(y) \in KO(\mathbb{C}P^{n-1})$ are polynomials in the appropriate truncated polynomial ring mentioned above for each $j = 1, 2, \ldots, d$.

So we would like to find polynomials c^j with integer coefficients and an integer s prime to p such that

$$(1 - \psi^k)\left(\sum_{j=1}^{d} c^j(y) \otimes \boldsymbol{U}_j\right) = sS \sum_{j=1}^{d} n_j \boldsymbol{U}_j \otimes y. \tag{3.4}$$

If τ denotes the inverse of the permutation determined by the action of $\psi_{\mathbb{C}}^k$ on $V_1, V_2, \ldots, V_d$ (and hence by the action of $\psi_{\mathbb{R}}^k$ on $\{U_1, \ldots, U_d\}$), the equation above can be written as

$$\sum_{j=1}^{d} c^j(y) \otimes \boldsymbol{U}_j - \sum_{j=1}^{d} \psi^k\big(c^{\tau(j)}(y)\big) \otimes \boldsymbol{U}_j = sS \sum_{j=1}^{d} n_j y \otimes \boldsymbol{U}_j.$$

Comparing the coefficients of each $\boldsymbol{U}_j$ we get

$$c^j(y) - \psi^k\big(c^{\tau(j)}(y)\big) = sSn_j y \quad (j = 1, 2, \ldots, d).$$

Setting $c^j(y) = \sum_{l=0}^{t'} c_l^j y^l$ for each $j = 1, 2, 3, \ldots, d$, we can write $\psi^k(c^{\tau(j)}(y))$ as $c_0^{\tau(j)} + \sum_{l=1}^{t'} c_l^{\tau(j)} \psi^k(y^l)$, and substituting the value of $\psi^k(y^l)$ from (3.2) we obtain

$$\sum_{l=0}^{t'} c_l^j y^l - \left(c_0^{\tau(j)} + \sum_{l=1}^{t'} c_l^{\tau(j)} \left(\sum_{\lambda=l}^{kl} R_k(\lambda, l) y^\lambda\right)\right) = sSn_j y.$$

Rearranging in powers of y we have

$$\sum_{r=0}^{t'} c_r^j y^r - \left(c_0^{\tau(j)} + \sum_{r=1}^{t'} \left(\sum_{l=\lceil \frac{r}{k} \rceil}^{r} R_k(r, l) c_l^{\tau(j)}\right) y^r\right) = sSn_j y.$$

Now, comparing this time the coefficients of each y^r, we get

$$c_0^j - c_0^{\tau(j)} = 0 \quad (1 \leqslant j \leqslant d), \tag{3.5}$$

$$c_1^j - c_1^{\tau(j)} R_k(1, 1) = sSn_j \quad (1 \leqslant j \leqslant d), \tag{3.6}$$

and for $2 \leqslant r \leqslant t'$,

$$c_r^j - \sum_{l=\lceil r/k \rceil}^{r} R_k(r, l) c_l^{\tau(j)} = 0, \quad 1 \leqslant j \leqslant d, \tag{3.7}$$

where the last equation corresponding to $r = 2m + 1$ is true only for (mod 2) when $n = 4m + 2$.

Clearly (3.5) holds if $c_0^j = 0$ for all $1 \leqslant j \leqslant d$. As for the Eqs. (3.6) and (3.7), we will show by induction on r that this system (considered as a linear system over the rational numbers) has solutions of the form

$$c_r^j = \frac{sS_r^t O_r^j(\mathcal{N}, \tau, p, k)}{p^{\nu_p(1-k^{2rd})}}, \quad 1 \leqslant j \leqslant d,\ 1 \leqslant r \leqslant t. \tag{3.8}$$

Consider Eq. (3.7). We see from the definition that $R_k(1,1) = k^2$, so for each $1 \leqslant j \leqslant d$ we have a system of d equations

$$\begin{aligned} c_1^j - k^2 c_1^{\tau(j)} &= sSn_j, \\ c_1^{\tau(j)} - k^2 c_1^{\tau^2(j)} &= sSn_{\tau(j)}, \\ &\vdots \\ c_1^{\tau^{d-1}(j)} - k^2 c_1^j &= sSn_{\tau^{d-1}(j)}, \end{aligned}$$

which is actually the same system for each j, except for the order of the equations. Multiplying the second equation by k^2, third one by k^4, and dth equation by $k^{2(d-1)}$, and adding all the equations, we get

$$(1-k^{2d})c_1^j = sS\sum_{l=0}^{d-1} n_{\tau^l(j)} k^{2l}$$

or simply

$$(1-k^{2d})c_1^j = sSO_1^j(\mathcal{N}, \tau, p, k).$$

Since $((1-k^{2d})/p^{\nu_p(1-k^{2d})})S_1^{t'} = S$, we have rational solutions of the form (3.8) for $r = 1$. Moreover, these are integers if and only if $O_1^j(\mathcal{N}, \tau, p, k)$ is a multiple of $p^{\nu_p(1-k^{2d})}$.

Assume for each $1 \leqslant l < r$ we have already obtained c_l^j in the form (3.8) for each $1 \leqslant j \leqslant d$. For each $2 \leqslant r \leqslant t$ and for each j with $1 \leqslant j \leqslant d$ we have the system

$$c_r^j - \sum_{l=\lceil r/k\rceil}^{r} R_k(r,l)c_l^{\tau(j)} = 0,$$

or, using $R_k(r,r) = k^{2r}$,

$$c_r^j - k^{2r}c_r^{\tau(j)} = \sum_{l=\lceil r/k\rceil}^{r-1} R_k(r,l)c_l^{\tau(j)}. \tag{3.9}$$

We proceed exactly as in the $r = 1$ case. We have d equations; the first equation is (3.9), the second one is the equation obtained by replacing j by $\tau(j)$, and continuing like this, the last equation is the equation obtained by replacing j in (3.9) by $\tau^{d-1}(j)$. Now, multiplying the second equation by k^{2r}, third one by k^{4r}, and dth equation by $k^{2r(d-1)}$, and adding all the equations we get

$$(1-k^{2dr})c_r^j = \sum_{l=\lceil r/k\rceil}^{r-1} R_k(r,l)\left(\sum_{i=1}^{d} c_l^{\tau^i(j)} k^{2r(i-1)}\right).$$

By substituting the values of $c_l^{\tau^i(j)}$ for $l < r$, denoting $O_l^j(\mathcal{N}, \tau, p, k)$ simply by O_l^j, and noting that $S_l^{t'} = S_l^{r-1} S_{r-1}^{t'}$, we can write

$$(1-k^{2dr})c_r^j = \sum_{l=\lceil r/k\rceil}^{r-1} R_k(r,l)\left(\sum_{i=1}^{d} s S_l^t \frac{O_l^{\tau^i(j)}}{p^{\nu_p(1-k^{2ld})}} k^{2r(i-1)}\right) = s S_{r-1}^{t'} O_r^j, \tag{3.10}$$

or, by the definition of $S_r^{t'}$,

$$c_r^j = \frac{s S_r^t O_r^j}{p^{\nu_p(1-k^{2rd})}}, \quad 1 \leqslant j \leqslant d,\ 1 \leqslant r \leqslant t.$$

So, the assertion follows when $n \neq 4m+2$.

Finally, in the case $n = 4m+2$ and $r = 2m+1$, the left hand side of Eq. (3.10) is zero (mod 2) since k is odd. If $p \neq 2$, the right hand side of (3.10) is also zero (mod 2) because s can be chosen as an even integer then. If $p = 2$, then s is odd, so $O_{2m+1}^j(\mathcal{N}, \tau, 2, 3)$ has to be even. □

Next, we shall prove the analogue of Theorem 3.6 for type-$\mathbb{C}$ orbits. To check that the two elements of $KO_G(\mathbb{C}P^{n-1}; \mathcal{U})$ in the complex part correspond to each other under $1-\psi_{\mathbb{R}}^k$, it suffices to check that they correspond under $1-\psi_{\mathbb{C}}^k$ in $KU(\mathbb{C}P^{n-1}) \otimes R(\mathbb{C})$, since ψ^k commutes with realification.

Let again G be a p-group, k be an odd generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$, and $V_1, V_2, \ldots, V_d$ be complex irreducible G modules of type $\mathbb{C}$ constituting some orbit of the action of $\psi_{\mathbb{C}}^k$. Let τ be the inverse of the permutation determined by the action of $\psi_{\mathbb{C}}^k$ in $V_1, V_2, \ldots, V_d$. We have

Theorem 3.7. *Let $\mathcal{U}$ be either the universe GR^∞ or $\mathcal{U}(M)$, and let $z = \mathrm{r}(\boldsymbol{N} \otimes (\xi - 1))$ where $\boldsymbol{N} = n_1 V_1 + \cdots + n_d V_d$ with $n_i \geqslant 0$ for each i. In the case $\mathcal{U} = \mathcal{U}(M)$ assume $V_1, V_2, \ldots, V_d$ have realifications in $\mathcal{U}(M)$. Let us write $\mathcal{N} = (n_1, \ldots, n_d)$. If z vanishes in $JO_G(\mathbb{C}P^{n-1}; \mathcal{U})_{(p)}$, then $p^{\nu_p(1-k^{rd})} \mid U_r^j(\mathcal{N}, \tau, p, k)$ for each $1 \leqslant j \leqslant d$ and for all $1 \leqslant r \leqslant n-1$. If G is an Abelian p-group these conditions are also sufficient.*

Proof. The proof is very similar to the proof of Theorem 3.6. Let $v = \xi - 1$ where ξ is the Hopf line bundle on $\mathbb{C}P^{n-1}$. By the remarks preceding the theorem, we find conditions for the existence of polynomials $c^j(v)$ in $KU(\mathbb{C}P^{n-1})$ with integer coefficients and an integer s prime to p such that

$$\left(1-\psi_{\mathbb{C}}^k\right)\left(\sum_{j=1}^{d} c^j(v) \otimes \boldsymbol{V}_j\right) = sT(\boldsymbol{N} \otimes_{\mathbb{C}} v), \tag{3.11}$$

where

$$T = T_0^{n-1} = \prod_{i=1}^{n-1} (1-k^{\mathrm{id}}) / \left(p^{\nu_p(1-k^{\mathrm{id}})}\right).$$

We recall that $\psi_{\mathbb{C}}^k(v) = (1+v)^k - 1$ (e.g., see [12, Ch. 4]) so that we have

$$\psi_{\mathbb{C}}^k(v^l) = \bigl((1+v)^k - 1\bigr)^l = \sum_{l}^{kl} Q_k(r,l)v^r \tag{3.12}$$

by the definition of $Q_k(r,l)$. Substituting $c^j(v) = \sum_{r=0}^n c_r^j v^r$ and the expression for $\psi_{\mathbb{C}}^k(v^l)$ from (3.12), proceeding exactly as in the previous proof and finally recalling that $Q_k(r,r) = k^r$ we get the equations

$$c_0^j - c_0^{\tau(j)} = 0, \quad 1 \leqslant j \leqslant d, \tag{3.13}$$

$$c_1^j - kc_1^{\tau(j)} = sTn_j, \quad 1 \leqslant j \leqslant d, \tag{3.14}$$

and for $2 \leqslant r \leqslant n-1$

$$c_r^j - k^r c^{\tau(j)} = \sum_{l=\lceil r/k \rceil}^{r-1} Q_k(r,l)c_l^{\tau(j)}, \quad 1 \leqslant j \leqslant d. \tag{3.15}$$

The rest of the proof is the same as the proof of Theorem 3.6. By induction on r we prove that this system has rational solutions

$$c_r^j = \frac{sT_r^{n-1}U_r^j}{p^{\nu_p(1-k^{rd})}}, \quad 1 \leqslant j \leqslant d,\ 1 \leqslant r \leqslant n-1,$$

where we have denoted $U_r^j(\mathcal{N}, \tau, p, k)$ simply by U_r^j, and the result follows. $\square$

4. Proofs of main theorems

Throughout this section G will be a finite group. As before, let H_p denote the smallest normal subgroup of H with p-primary index for $H < G$ and for any prime p, and let $\Phi(p)$ denote the set $\{H < G\colon (|WH|, p) = 1\}$. First we recall certain notions. See Sections 8 and 9 of [10] for more details.

Let $\mathcal{U}$ be a G-universe, h be a multiplicative $\mathcal{U}$-graded G-cohomology theory and let E be a G vector bundle over a space X with trivial G-action such that $E_x = N$ for some $N \subset \mathcal{U}$. Given a class $\nu \in h_G^N(TE)$ we write ν_x for its image under the composite map

$$h_G^N(TE) \xrightarrow{i_x^*} h_G^N(TE_x) \underset{\cong}{\xrightarrow{\iota_x}} h_G^N(S^N) \cong h_G^0(S^0),$$

where TE denotes the Thom Space. We say that ν is an h_G^*-orientation for E in dimension N if each ν_x is a unit in $h_G^0(S^0)$.

We also recall that if $A(G;\mathcal{U})$ is the Burnside ring $\operatorname{colim}_{V\subset\mathcal{U}}[S^V, S^V]_G$, $H < G$ and p a prime, then $q(H;\, p;\mathcal{U})$ is the set of elements of $A(G;\mathcal{U})$ represented by maps $f: S^V \to S^V$ such that H-fixed point map f^H has degree divisible by p. The set $q(H;\, p;\mathcal{U})$ is a prime ideal of $A(G;\mathcal{U})$ and all prime ideals of $A(G;\mathcal{U})$ are of this form.

Slightly modifying the definition in [10], we call a universe $\mathcal{U}$ excisive if, for each prime p dividing the order of G and for every $K \in \Phi(p)$,

(i) both K and $K_p \in \operatorname{Iso}(\mathcal{U})$,

(ii) the restriction map

$$i^*: h^*_G(X)_{q(K,p;\mathcal{U})} \to h^*_G\big(X^{(K_p)}\big)_{q(K,p;\mathcal{U})}$$

is an isomorphism [10] for any G-CW complex X and any $\mathcal{U}$-graded cohomology theory h^*_G, where $X^{(K_p)} = G \cdot X^{K_p}$.

With this terminology at hand we have the following lemma and theorems which are the p-primary analogues of Theorem 10.1 and Lemma 10.2 of [10]. Since their proofs are the same we shall omit them. Note that for GR^∞ these are proved in [9], and Namboodiri made necessary modifications for a general universe $\mathcal{U}$, for the 2-primary case. We take M as a unitary G-module, ξ as the Hopf line bundle over $\mathbb{C}P^{n-1}$ as before.

Theorem 4.1. *Let p be any prime. If there exists an $\omega^*_H(?)_{q(H,p;\mathcal{U}(M))}$-orientation for each $H \in \Phi(p)$ in dimension $r(M)$, then $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in the group $JO_G(\mathbb{C}P^{n-1}; \mathcal{U}(M))_{(p)}$.*

Lemma 4.2. *Suppose that $\mathcal{U}(M)$ is an excisive universe and p is a prime dividing the order of G. Let $H \in \Phi(p)$, and assume that $\mathrm{r}(\boldsymbol{M}^{H_p} \otimes (\xi - 1))$ vanishes in $JO_{H/H_p}(\mathbb{C}P^{n-1}; \mathcal{U}(M^{H_p}))_{(p)}$. Then there exists an $\omega^*_H(?)_{q(H,p;\mathcal{U}(M))}$-orientation for $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ in dimension $r(M)$.*

The reason for the excisive universe hypothesis is the failure of the generalized excision result Theorem 8.8 of [10]. In the case that $|G|$ is prime to p, the excision result Theorem 8.5 of [10] holds, and by a p-primary analogue of Lemma 9.3 of [10], the existence of an $\omega^*_H(?)_{q(H,p;\mathcal{U}(M))}$-orientation for each $H < G$ follows without any excisive universe hypothesis. The obvious p-primary analogues of Lemmas 9.4–9.6 of [10] with merely the prime 2 replaced with p, and $\mathbb{R}P^n$ replaced with $\mathbb{C}P^n$, lead to the following analogue of Theorem 9.1 of [10].

Theorem 4.3. *Let the order of G be prime to p. Then $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in the group $JO_G(\mathbb{C}P^{n-1}; \mathcal{U}(M))_{(p)}$ if and only if $\mathrm{r}(\boldsymbol{M}^H \otimes (\xi - 1))$ vanishes in $JO(\mathbb{C}P^{n-1})_{(p)}$ for each $H < G$.*

Proof of Theorem 1.1. Assume that there exists a G-cross section of $W_n(M)$. Then by Theorem 2.1, $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in $JO_G(\mathbb{C}P^{n-1})$. Hence, we deduce from Theorems 3.6, 3.7 applied in the case $\mathcal{U} = GR^\infty$ that (i) and (ii) of Theorem 1.1 hold. On the other hand, vanishing of $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ in $JO_G(\mathbb{C}P^{n-1})$ implies the vanishing of $\mathrm{r}(\boldsymbol{M}^H \otimes (\xi - 1))$ in $JO_G(\mathbb{C}P^{n-1})$ for any $H < G$, so condition (iii) of Theorem 1.1 also holds.

Conversely, let G be an Abelian p-group, and let the conditions (i)–(iii) of Theorem 1.1 hold. By the definition of $\mathcal{U}(M)$, whenever $m_{i,j} > 0$, the realification of the corresponding irreducible G-module $V_{i,j}$ is a submodule of $\mathcal{U}(M)$. So from Theorems 3.6 and 3.7, applied in the case $\mathcal{U} = \mathcal{U}(M)$, we deduce that $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in the group $JO_G(\mathbb{C}P^{n-1}; \mathcal{U}(M))_{(p)}$. On the other hand, condition (iii) of Theorem 1.1, with the help of Theorem 4.3, implies that $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in $JO_G(\mathbb{C}P^{n-1}; \mathcal{U}(M))_{(q)}$ for $p \neq q$ also. Therefore $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in $JO_G(\mathbb{C}P^{n-1}; \mathcal{U}(M))$. Now, by Theorem 2.1, $W_n(M)$ has a G-equivariant cross section. □

Proof of Theorem 1.2. If $W_n(M)$ admits a G-cross section, then $W_n(M^{H_p})$ obviously admits a H/H_p cross section for any prime dividing the order of G and for any $H \in \Phi(p)$. The converse follows from Theorem 4.1, Lemma 4.2, Theorems 4.3 and 2.1. □

Proof of Theorem 1.3. Since $\dim_{\mathbb{R}} M^G \geqslant 4n - 2$ for $n > 1$, there exists a G-equivariant $(2n - 1)$-frame field whose complement in $T(S(M))$ admits a G-equivariant complex structure if $\mathrm{r}(\boldsymbol{M} \otimes (\xi - 1))$ vanishes in $JO_G(\mathbb{C}P^{n-1}; \mathcal{U}(M))_{(2)}$ for any prime dividing the order of G and for any $H \in \Phi(p)$ by Theorem 4.1 of [11]. Now the result follows from Theorems 3.6 and 3.7. □

Proof of Theorem 1.4. Necessity is clear from the definitions upon restrictions to fixed point sets, and the converse follows from Lemma 4.2, Theorem 4.1 above and Theorem 4.1 of [11]. □

5. An example

Let $G = \mathbb{Z}/4\mathbb{Z}$ and let M be a unitary G-module. We shall find sufficient and necessary conditions for $W_3(M)$ to admit an equivariant G-cross section.

The group $\mathbb{Z}/4\mathbb{Z}$ has four irreducible one-dimensional unitary representations V_0, V_0^-, V_1, V_2. In these representations, the generator $g = 1 \bmod 4$ acts as multiplication by $1, -1$, $\mathrm{e}^{\mathrm{i}(\pi/2)} = \mathrm{i}$, $\mathrm{e}^{\mathrm{i}(3\pi/2)} = -\mathrm{i}$, respectively [5]. We shall apply Theorem 1.1 when $p = 2$ and $k = 3$. Since all irreducible representations of $\mathbb{Z}/4\mathbb{Z}$ are one-dimensional, $\psi^3_{\mathbb{C}}(V) = V^3$ for every irreducible module V given above. Hence we have $\psi^3_{\mathbb{C}}(V_0) = V_0$, $\psi^3_{\mathbb{C}}(V_0^-) = V_0^-$, $\psi^3_{\mathbb{C}}(V_1) = V_2$, $\psi^3_{\mathbb{C}}(V_2) = V_1$. So the orbits of the action of $\psi^3_{\mathbb{C}}$ are $\mathcal{O}_1 = \{V_0\}$, $\mathcal{O}_2 = \{V_0^-\}$ which are of real type and $\mathcal{O}_3 = \{V_1, V_2\}$ which is of complex type.

Let $M = m_0 V_0 + m_1 V_0^- + m_1 V_1 + m_2 V_2$. Then we have $\mathcal{M}_1 = (m_{1,1}) = (m_0)$, $\mathcal{M}_2 = (m_{2,1}) = (m_0^-)$, $\mathcal{M}_3 = (m_{3,1}, m_{3,2}) = (m_1, m_2)$ in the notation of Theorem 1.1. Thus, we have $s_1 = 2, s_2 = 1, d_1 = 1, d_2 = 1, d_3 = 2, t = 1, n = 3$, and $\tau_1 = (1), \tau_2 = (1), \tau_3 = (1, 2)$ are the respective permutations (in cycle notation) for the orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$. We compute the following numbers

$$O_1^1(\mathcal{M}_1, \tau_1, 2, 3) = \sum_{l=0}^{d_1 - 1 = 0} m_{1, \tau_1^l(1)} 3^{2l} = m_{1,1} = m_0,$$

$$O_1^1(\mathcal{M}_2, \tau_2, 2, 3) = \sum_{l=0}^{d_2 - 1 = 0} m_{2, \tau_2^l(1)} 3^{2l} = m_{2,1} = m_0^-,$$

$$U_1^1(\mathcal{M}_3, \tau_3, 2, 3) = \sum_{l=0}^{d_3 - 1 = 1} m_{3, \tau_3^l(1)} = m_{3,1} + 3m_{3,2} = m_1 + 3m_2,$$

$$U_1^2(\mathcal{M}_3,\tau_3,2,3)=\sum_{l=0}^{d_3-1=1} m_{3,\tau_3^l(2)}=m_{3,2}+3m_{3,1}=m_2+3m_1,$$

$$\begin{aligned}&U_2^j(\mathcal{M}_3,\tau,2,3)\\&\quad=\sum_{l=\lceil 2/3\rceil=1}^{r-1=1} Q_3(2,l)T_l^1\left(\frac{U_l^{\tau_3^1(j)}(\mathcal{M}_3,\tau_3,2,3)}{2^{\nu_2(1-3^{2l})}}+\frac{U_l^{\tau_3^2(j)}(\mathcal{M}_3,\tau_3,2,3)}{2^{\nu_2(1-3^{2l})}}3^2\right).\end{aligned}$$

Since $Q_3(2,1)=3$ (the coefficient of x^2 in $(1+x)^3-1$), and $T_1^1=1$, omitting the argument$(\mathcal{M}_3,\tau_3,2,3)$ in all terms, we get

$$\begin{aligned}U_2^1&=\tfrac{3}{8}(U_1^2+9U_1^1),\\U_2^2&=\tfrac{3}{8}(U_1^1+9U_1^2).\end{aligned}$$

Substituting the values of U_1^1 and U_1^2 we obtain

$$\begin{aligned}U_2^1&=\tfrac{3}{2}(3m_1+7m_2),\\U_2^2&=\tfrac{3}{2}(7m_1+3m_2).\end{aligned}$$

On the other hand, $\mathbb{Z}/4\mathbb{Z}$ has four subgroups $\{0\}$, $\mathbb{Z}/4\mathbb{Z}$, $H=\{0,2g\}$. We have $\dim_{\mathbb{C}} M^{\{0\}}=m_0+m_0^-+m_1+m_2$, $\dim_{\mathbb{C}} M^G=m_0$, $\dim_{\mathbb{C}} M^H=m_0+m_0^-$. Finally, we have $b_3=24$, and the only prime $q\neq 2$ dividing this number is $q=3$.

Now, by Theorem 1.1, $W_3(M)$ admits a G-cross section if and only if all of the following conditions hold.

$$\begin{aligned}&2^{\nu_2(1-3^{2d_1})}=2^3\mid m_0, && 2^{\nu_2(1-3^{2d_2})}=2^3\mid m_0^-,\\&2^{\nu_2(1-3^{d_3})}=2^3\mid m_1+3m_2, && 2^{\nu_2(1-3^{d_3})}=2^3\mid 3m_1+m_2,\\&2^{\nu_2(1-3^{2d_3})}=2^5\mid \frac{3}{2}(3m_1+7m_2), && 2^{\nu_2(1-3^{2d_3})}=2^5\mid \frac{3}{2}(7m_1+3m_2),\\&q^{\nu_q(b_3)}=3\mid \dim_{\mathbb{C}} M^{\{0\}}\\&\qquad=m_0+m_0^-+m_1+m_2, && q^{\nu_q(b_3)}=3\mid \dim_{\mathbb{C}} M^G=m_0,\\&q^{\nu_q(b_3)}=3\mid \dim_{\mathbb{C}} M^H=m_0+m_0^-.\end{aligned}$$

A simple computation shows that these conditions are equivalent to

$$24\mid m_0,\qquad 24\mid m_0^-,\qquad 96\mid m_1+m_2,\qquad 64\mid m_1+5m_2.$$

Note that $\dim_{\mathbb{R}} M^G=2m_0>4n-2=10$ when these conditions hold, so that the hypothesis for the converse of the Theorem 1.1 is satisfied.

By Theorem 1.3, the same computations also show that there are $2n-1=5$ real frame fields on $S(M)$ whose complement admits a G-equivariant complex structure if the following conditions hold.

$$8\mid m_0,\qquad 8\mid m_0^-,\qquad 32\mid m_1+m_2,\qquad 64\mid m_1+5m_2.$$

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